* Fracture *

HW-1 — Show that $\sigma_x = \dfrac{\partial^2 \psi}{\partial y^2} = \text{Re}\, Z - y\, \text{Im}\, Z'$

$$\sigma_y = \frac{\partial^2 \psi}{\partial x^2} = \text{Re}\, Z + y\, \text{Im}\, Z'$$

$$\tau_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y} = -y\, \text{Re}\, Z'$$

$$\psi = \text{Re}\,(\bar{\bar{Z}}) + y\, \text{Im}\,(\bar{Z})$$ (Westergaard's function)

⇒ CR equations →

$$* \quad \frac{\partial(\text{Re}\, Z)}{\partial x} = \frac{\partial(\text{Im}\, Z)}{\partial y} = \text{Re}\left(\frac{\partial Z}{\partial z}\right)$$

$$* \quad \frac{\partial(\text{Im}\, Z)}{\partial x} = -\frac{\partial(\text{Re}\, Z)}{\partial y} = \text{Im}\left(\frac{\partial Z}{\partial z}\right)$$

— (A)

$$\therefore \quad \psi = \text{Re}(\bar{\bar{Z}}) + y\, \text{Im}\,(\bar{Z})$$

$$\bar{Z} = \frac{d\bar{\bar{Z}}}{dz}, \quad Z = \frac{d\bar{Z}}{dz}, \quad Z' = \frac{dZ}{dz}$$

— (B) Z is analytic function

$$\therefore \quad \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(\text{Re}\, \bar{\bar{Z}}) + y \frac{\partial}{\partial x}(\text{Im}\, \bar{Z})$$ Using CR eq^{ns} (A) ⇒

$$= \text{Re}\left(\frac{\partial \bar{\bar{Z}}}{\partial z}\right) + y\, \text{Im}\left(\frac{\partial \bar{Z}}{\partial z}\right)$$ --- using (B) ⇒

$$\frac{\partial \psi}{\partial x} = \text{Re}\,(\bar{Z}) + y\, \text{Im}(Z)$$

$$\therefore \quad \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}(\text{Re}\, \bar{Z}) + y \frac{\partial}{\partial x}(\text{Im}\, Z)$$ -- Using (A) ⇒

$$\frac{\partial^2\psi}{\partial x^2} = Re\left(\frac{\partial \bar{Z}}{\partial z}\right) + y\,Im\left(\frac{\partial Z}{\partial z}\right) \quad \ldots \text{Using } (B) \Rightarrow$$

$$\boxed{\frac{\partial^2\psi}{\partial x^2} = Re(Z) + y\,Im(Z') = \sigma_y}$$

$$* \quad \psi = Re(\bar{\bar{Z}}) + y\,Im(\bar{Z})$$

$$\frac{\partial\psi}{\partial y} = \frac{\partial}{\partial y}(Re\,\bar{\bar{Z}}) + y\frac{\partial}{\partial y}(Im\,\bar{Z}) + Im(\bar{Z})\frac{\partial}{\partial y}(y) \ldots \text{Using } (A)$$

$$= -Im\left(\frac{\partial \bar{\bar{Z}}}{\partial z}\right) + y\,Re\left(\frac{\partial \bar{Z}}{\partial z}\right) + Im(\bar{Z}) \ldots \text{Using } (B) \Rightarrow$$

$$= -Im(\bar{Z}) + y\,Re(Z) + Im(\bar{Z})$$

$$= y\,Re(Z)$$

$$\frac{\partial^2\psi}{\partial y^2} = y\frac{\partial}{\partial y}(Re\,Z) + Re(Z)\frac{\partial}{\partial y}(y) \ldots \text{Using } (A) \Rightarrow$$

$$= -y\,Im\left(\frac{\partial Z}{\partial z}\right) + Re(Z) \quad \ldots \text{Using } (B) \Rightarrow$$

$$\boxed{\frac{\partial^2\psi}{\partial y^2} = Re(Z) - y\,Im(Z') = \sigma_x}$$

$$\frac{\partial\psi}{\partial x\partial y} = y\frac{\partial}{\partial x}(Re\,Z)$$

$$= y\,Re\left(\frac{\partial Z}{\partial z}\right)$$

$$\therefore \boxed{\tau_{xy} = \frac{-\partial\psi}{\partial x\partial y} = -y\,Re(Z')}$$