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HOME - WORK

Sub: Mechanical Behavior of Materials.

Topic: To locate the Von Mises' Criterion in 3D; with considering principle axises as x, y and z-axis.

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Q. Principle stress form of the ~~[illegible]~~; Von-Mises' Criterion

$$\frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \geqslant K^2$$

with considering σ_1, σ_2, and σ_3 as x-axis, y-axis and z-axis respectivelly, generate the curve in '3D'.

Solution:

Given equation,

$$\frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \geqslant K^2$$

Let $\sigma_1 = x$; $\sigma_2 = y$; $\sigma_3 = z$

than corresponding equation will be in the following form,

$$\frac{1}{6}\left[(x-y)^2+(y-z)^2+(z-x)^2\right] = K^2$$

$$\therefore (x-y)^2+(y-z)^2+(z-x)^2 = 6K^2$$

$$x^2+y^2-2xy+y^2+z^2-2yz+z^2+x^2-2zx = 6K^2$$

or, $$2x^2+2y^2+2x^2-2xy-2yz-2zx = 6K^2$$

$$\therefore x^2+y^2+z^2-xy-yz-zx = 3K^2 \quad \text{——— (1)}$$

for locating the curve presenting the equation (1), in 3D, we will do this in two steps.

1) First we will ~~find that the~~ check that the given curve will present of which quardratic curve.

2) we will locate the curve in 3D, with

i) ~~Aa~~ Standard form of the quardratic surface,

$$ax^2+by^2+cz^2+2hxy+2fyz+2gzx+2px+2qy+2rz+d=0 \quad (2)$$

for this curve, two matrix are define, and one equation.

corresponding to the rank of the matrices and solution of the equation, define different type of surfaces in 3D.

$$e=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \; ; \; E=\begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{bmatrix}$$

Refer to
The Elements Of Coordinate Geometry by S. L. Loney
for these steps

and equation is

$$\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0$$; roots of this equation are K_1, K_2, and K_3.

Now, for equation (1);

$$2x^2+2y^2+2z^2-2xy-2yz-2zx-6K^2=0.$$

comparing this equation with equation (2), we get

$a=2$	$2h=-2$; $h=-1$	$p=0$	
$b=2$	$2f=-2$; $f=-1$	$q=0$	$d=-6K^2$.
$c=2$	$2g=-2$; $g=-1$	$r=0$	

So, $e = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$\therefore$ rank of $e = 2$

$$E = \begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & -6K^2 \end{bmatrix}$$

rank of $E = 3$

Now, the equation is

$$\therefore \begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\{(2-\lambda)^2-1\} - 1\{1+2-\lambda\} - 1\{1+2-\lambda\} = 0$

$\Rightarrow (2-\lambda)\{4+\lambda^2-4\lambda-1\} - 1\{3-\lambda\} - 1\{3-\lambda\} = 0.$

$\Rightarrow (2-\lambda)(\lambda^2-4\lambda+3) - 3 + \lambda - 3 + \lambda = 0$

$\Rightarrow 2\lambda^2 - 8\lambda + \cancel{6} - \lambda^3 + 4\lambda^2 - 3\lambda - \cancel{6} + 2\lambda = 0$

$\Rightarrow -(\lambda^3 - 6\lambda^2 + 9\lambda) = 0.$

as non-zero values of the λ have same sign.

So, rank of $e = 2$

rank of $E = 3$; and λ values show that the given ~~curve~~ ~~with~~ quardratic surface is cylinder.

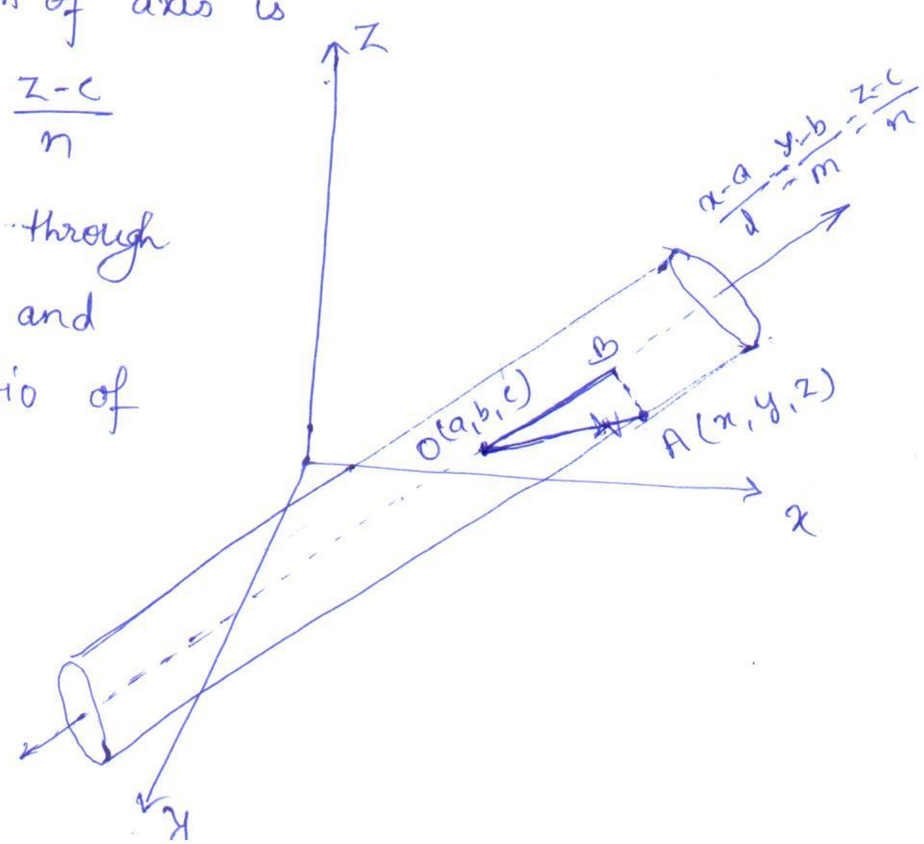
[Ref :: http://mathworld.wolfram.com/QuadraticSurface.html

2) For locating a cylinder in 3D, we need to know the axis equation (a straight line) and the perpendicular distance from the surface of cylinder to their axis.

Now, let equation of axis is

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

this line passes through the point (a, b, c) and have direction ratio of (l, m, n).



$\therefore$ Direction cosine of the line $= \left(\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}} \right)$

consider a variable point $A(x, y, z)$ on the cylinder. and draw a perpendicular 'AB' on the line of axis.

OB = projection of OA on the line (axis of cylinder)

AB = Perpendicular distance of line and surface of cylinder $= h$ (let).

$$OA^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \quad \{\because \text{by point-point distance formula}\}$$

$$OB = \frac{(x-a)l}{\sqrt{l^2+m^2+n^2}} + \frac{(y-b)m}{\sqrt{l^2+m^2+n^2}} + \frac{(z-c)n}{\sqrt{l^2+m^2+n^2}}$$

$$OB = \frac{1}{\sqrt{l^2+m^2+n^2}} \{(x-a)l + (y-b)m + (z-c)n\}$$

$$OB^2 = \frac{1}{l^2+m^2+n^2} \{(x-a)l + (y-b)m + (z-c)n\}^2$$

Now, In $\triangle OAB$;

By pythagorous theorem,

$$OA^2 = OB^2 + AB^2$$

$$\Rightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 = \frac{1}{l^2+m^2+n^2} \{(x-a)l + (y-b)m + (z-c)n\}^2 + h^2$$

$$(l^2+m^2+n^2)\{(x-a)^2+(y-b)^2+(z-c)^2\} = \{(x-a)l+(y-b)m+(z-c)n\}^2 + h^2$$

$$(l^2+m^2+n^2)\{(x-a)^2+(y-b)^2+(z-c)^2\} = (x-a)^2l^2+(y-b)^2m^2+(z-c)^2n^2+2lm(x-a)(y-b)$$
$$+2mn(y-b)(z-c)+2nl(z-c)(x-a)+h^2$$

$$(m^2+n^2)(x-a)^2+(l^2+n^2)(y-b)^2+(m^2+n^2)(z-c)^2-2lm(xy-ay-bx+ab)-2mn(yz-bz-cy+bc)$$
$$-2nl(zx-az-cx+ca)-h^2=0$$

$$(m^2+n^2)(x^2-2ax+a^2)+(l^2+m^2)(y^2-2by+b^2)+(m^2+n^2)(z^2-2cz+c^2)-2lm(xy-ay-bx+ab)$$
$$-2mn(yz-bz-cy+bc)-2nl(zx-az-cx+ca)-h^2=0$$

$$(m^2+n^2)x^2+(l^2+m^2)y^2+(m^2+n^2)z^2-2lm\cdot xy-2mn\cdot yz-2nl\cdot zx-x\{2a(n^2+m^2)-2lm\cdot b-2nl\cdot c\}$$
$$-y\{2b(l^2+m^2)-2lm\cdot a-2mn\cdot c\}-z\{2c(m^2+l^2)-2mn\cdot b-2nl\cdot a\}$$
$$+a^2(m^2+n^2)+b^2(l^2+m^2)+c^2(m^2+l^2)-2lm\cdot ab-2mn\cdot bc-2nl\cdot ca-h^2=0$$

Now equating the cofficient of this equation with the following give equation

$$x^2+y^2+z^2-xy-yz-zx-3K^2=0 \quad \{\text{here actually } x=\sigma_1;\ y=\sigma_2;\ z=\sigma_3\}.$$

$$\left.\begin{aligned} l^2+m^2 &= 1 \\ m^2+n^2 &= 1 \\ n^2+l^2 &= 1 \end{aligned}\right\}$$ on solving this equation, we get

$$l = \frac{1}{\sqrt{2}}, \quad m = \frac{1}{\sqrt{2}}, \quad n = \frac{1}{\sqrt{2}}$$

$$2lm = 1$$
$$2mn = 1$$
$$2nl = 1$$

So, direction cosines (given on page 5, first line) will becomes [1/sqrt(3), 1/sqrt(3),1/sqrt(3)]

$$2a(n^2+m^2) - 2lm \cdot b - 2nl \cdot c = 0 \quad \text{———} \quad (1)$$
$$2b(l^2+n^2) - 2lm \cdot a - 2mn \cdot c = 0 \quad \text{———} \quad (2)$$
$$2c(l^2+m^2) - 2mn \cdot b - 2nl \cdot a = 0 \quad \text{———} \quad (3)$$

$$a^2(m^2+n^2) + b^2(l^2+n^2) + c^2(m^2+l^2) - 2lm \cdot ab - 2mn \cdot bc - 2nl \cdot ca - h^2 = 3K^2 \quad \text{———} \quad (7)$$

Now, putting the value of l, m, and n in equations ①, ② and ③, we have

$$2a - b - c = 0$$
$$2b - a - c = 0$$
$$2c - b - a = 0$$

$$\Rightarrow \quad 2a - b - c = 0 \quad \text{———} \quad (4)$$
$$-a + 2b - c = 0 \quad \text{———} \quad (5)$$
$$-a - b + 2c = 0 \quad \text{———} \quad (6)$$

for equation ⑤ and ⑥

$$\frac{a}{4-1} = \frac{b}{1+2} = \frac{c}{1+2} = \omega \text{ (Let)}$$

$$\therefore a = 3\omega, \quad b = 3\omega, \quad c = 3\omega$$

Now, on substituting a, b, c values in equation ④, we get
$$\omega = 0$$

Substituting the values of a, b, c, l, m and n in equⁿ (7) we get,

$$0 + 0 + 0 - 0 - 0 - 0 - h^2 = -3K^2$$

$$\therefore h = \sqrt{3}\,K \quad \left\{ h \neq -\sqrt{3}\,K \text{, because distance is always positive quantity} \right\}.$$

Now equation of axis of the cylinder is

$$\frac{x-0}{1/\sqrt{2}} = \frac{y-0}{1/\sqrt{2}} = \frac{z-0}{1/\sqrt{2}}$$

So, the axis of cylinder passes through origin and is at equal angle to all coordinate axes

and perpendicular distance from the surface of cylinder to the axis is $\sqrt{3}\,K$.

So, the radius of cylinder is sigma_ys

Now location of the cylinder with respect to the σ_1, σ_2 and σ_3 axis will look like following shown surface