MT253:Extra Credit Problem

Kamalnath K

August 28, 2015

1)Show that orthotropic, tetragonal and cubic systems will have only 9, 7 and 3, respectively, independent elastic moduli.

- Stiffness tensor transforms as $C'_{ijkl} = \sum a_{ip}a_{jq}a_{kr}a_{ls}C_{pqrs}$.
- If crystal system is symmetric with respect to transformation A then

$$C'_{ijkl} = \sum a_{ip}a_{jq}a_{kr}a_{ls}C_{pqrs} = C_{ijkl} \tag{1}$$

ORTHORHOMBIC: Essential symmetry is the existence of three perpendicular 2-fold axis of symmetry. Let A_i represent the 2-fold rotation about the i^{th} axis. $i = 1, 2, 3$. Then

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In all these transformation matrices, we see that $a_{ij} = 0$ if $i \neq j$. Hence, in (1), only one term survives after the summation. Thus

$$C'_{ijkl} = a_{ii}a_{jj}a_{kk}a_{ll}C_{ijkl} = C_{ijkl} \tag{2}$$

Thus if C_{ijkl} is non-zero , then $a_{ii}a_{jj}a_{kk}a_{ll} = 1$. In other words, if $a_{ii}a_{jj}a_{kk}a_{ll} = -1$ then $C_{ijkl} = 0$. Now looking for the possibilities of -1 we have

$$C'_{14} = C'_{1123} \to \underbrace{a_{11}a_{11}a_{22}a_{33}}_{A \to A_3} = -1 \quad \Rightarrow C_{1123} = 0 \tag{3}$$

$$C'_{15} = C'_{1113} \to \underbrace{a_{11}a_{11}a_{11}a_{33}}_{A \to A_3} = -1 \quad \Rightarrow C_{1113} = 0 \tag{4}$$

$$C'_{16} = C'_{1112} \to \underbrace{a_{11}a_{11}a_{11}a_{22}}_{A \to A_2} = -1 \quad \Rightarrow C_{1112} = 0 \tag{5}$$

$$C'_{24} = C'_{2223} \to \underbrace{a_{22}a_{22}a_{22}a_{33}}_{A \to A_3} = -1 \quad \Rightarrow C_{2223} = 0 \tag{6}$$

$$C'_{25} = C'_{2213} \to \underbrace{a_{22}a_{22}a_{11}a_{33}}_{A \to A_3} = -1 \quad \Rightarrow C_{2213} = 0 \tag{7}$$

$$C'_{26} = C'_{2212} \to \underbrace{a_{22}a_{22}a_{11}a_{22}}_{A \to A_2} = -1 \quad \Rightarrow C_{2212} = 0 \tag{8}$$

$$C'_{34} = C'_{3323} \to \underbrace{a_{33}a_{33}a_{22}a_{33}}_{A \to A_3} = -1 \quad \Rightarrow C_{3323} = 0 \tag{9}$$

$$C'_{35} = C'_{3313} \to \underbrace{a_{33}a_{33}a_{11}a_{33}}_{A \to A_3} = -1 \quad \Rightarrow C_{3313} = 0 \tag{10}$$

$$C'_{36} = C'_{3312} \to \underbrace{a_{33}a_{33}a_{11}a_{22}}_{A \to A_1} = -1 \quad \Rightarrow C_{3312} = 0 \tag{11}$$

$$C'_{45} = C'_{2313} \to \underbrace{a_{22}a_{33}a_{11}a_{33}}_{A \to A_1} = -1 \quad \Rightarrow C_{2313} = 0 \tag{12}$$

$$C'_{46} = C'_{2312} \to \underbrace{a_{22}a_{33}a_{11}a_{22}}_{A \to A_3} = -1 \quad \Rightarrow C_{2312} = 0 \tag{13}$$

$$C'_{56} = C'_{1312} \to \underbrace{a_{11}a_{33}a_{11}a_{22}}_{A \to A_2} = -1 \quad \Rightarrow C_{1312} = 0 \tag{14}$$

Thus, 12 elastic modulli became zero. Initially we had 21 independent constants. So, now we have $21 - 12 = 9$ independent constants.

TETRAGONAL: Essential symmetry is the existence of one 4-fold(90^o) rotation axis. Let A represent the 4-fold rotation with respect to 3-axis.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As a 4-fold axis has a two-fold inside, we would be able to use some of the equations of orthogonal system will also hold now. In the last section, all the equations derived from the

A_3 will also hold here. Thus,

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = 0$$

which gives 7 constraints. In the current A matrix, it should be noted that each row has only one non-zero entry. Thus, for a given $ijkl$, there will be only one non-zero term in the summation of (1). Hence, this case also can be done easily as follows,

$$\begin{aligned}
C'_{11} &= C'_{1111} = a_{12}a_{12}a_{12}a_{12}C_{2222} = C_{2222} = C_{22} && = C_{11} \quad (15)\\
C'_{13} &= C'_{1133} = a_{12}a_{12}a_{33}a_{33}C_{2233} = C_{2233} = C_{23} && = C_{13} \quad (16)\\
C'_{44} &= C'_{2323} = a_{21}a_{33}a_{21}a_{33}C_{1313} = C_{1313} = C_{55} && = C_{44} \quad (17)
\end{aligned}$$

$$\boxed{\begin{aligned}
C'_{16} &= C'_{1112} = a_{12}a_{12}a_{12}a_{21}C_{2221} = -C_{2221} = -C_{26} \\
C'_{26} &= C'_{2212} = a_{21}a_{21}a_{12}a_{21}C_{1121} = -C_{1121} = -C_{16}
\end{aligned}}
\begin{aligned}
&= C_{16} \quad (18)\\
&= C_{26} \quad (19)
\end{aligned}$$

equations for 2 moduli - so counted as only one equation

$$\begin{aligned}
C'_{36} &= C'_{3312} = a_{33}a_{33}a_{12}a_{21}C_{3321} = -C_{3312} = -C_{36} && = C_{36} \quad (20)\\
C'_{45} &= C'_{2313} = a_{21}a_{33}a_{12}a_{33}C_{1323} = -C_{1323} = -C_{54} = -C_{45} && = C_{45} \quad (21)\\
C'_{56} &= C'_{1312} = a_{12}a_{33}a_{12}a_{21}C_{2321} = -C_{2321} = -C_{46} && = C_{56} \quad (22)
\end{aligned}$$

As we can see (18) and (19) are dependent which actually give one constraint. From (20), $C_{36} = 0$. From (21), $C_{45} = 0$. As $C_{46} = 0$, from (22), $C_{56} = 0$.
Thus the number of independent modulli is

$$21 - \underbrace{7}_{\text{From equation in this session}} - \underbrace{7}_{\text{From equations in the last session}} = 7$$

CUBE: Essential symmetry is the existence of 4 3-fold axis.Let A_i be the rotation matrix such that

- $A_1 \rightarrow$ rotation along [1 1 1] direction.
- $A_2 \rightarrow$ rotation along [-1 1 1] direction.
- $A_3 \rightarrow$ rotation along [1 -1 1] direction.
- $A_4 \rightarrow$ rotation along [-1 -1 1] direction.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

Here also, in each row of A_i has only one non-zero entry. Hence, in the summation of (1) only one term will remain.
Considering $A \to A_1$m we get the following independent equations.

$$C'_{11} = C'_{1111} = a_{12}a_{12}a_{12}a_{12}C_{2222} = C_{2222} = C_{22} = C_{11} \tag{23}$$
$$C'_{22} = C'_{2222} = a_{23}a_{23}a_{23}a_{23}C_{3333} = C_{3333} = C_{33} = C_{22} \tag{24}$$
$$C'_{12} = C'_{1122} = a_{12}a_{12}a_{23}a_{23}C_{2233} = C_{2233} = C_{23} = C_{12} \tag{25}$$
$$C'_{23} = C'_{2233} = a_{23}a_{23}a_{31}a_{31}C_{3311} = C_{3311} = C_{31} = C_{23} \tag{26}$$
$$C'_{44} = C'_{2323} = a_{23}a_{31}a_{23}a_{31}C_{3131} = C_{3131} = C_{55} = C_{44} \tag{27}$$
$$C'_{55} = C'_{1313} = a_{12}a_{31}a_{12}a_{31}C_{2121} = C_{2121} = C_{66} = C_{55} \tag{28}$$

As the cubic system is a very symmetric object, It has the symmetry elements of few lower classes also. In particular, it satisfies the requirement of orthorhombic system. We know if a system is symmetric with respect to B_1, B_2 then it is also symmetric with respect to $B_1 * B_2$. If $B_1\vec{X} = \vec{X}$ and $B_2\vec{X} = \vec{X}$ then

$$(B_1 * B_2)\vec{X} = B_1 * (B_2\vec{X}) = B_1\vec{X} = \vec{X}$$

We also notice that

$$A_3A_1A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (A_1)_{\text{orthorhombic}}$$

$$A_2A_3A_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = (A_2)_{\text{orthorhombic}}$$

$$A_1A_2A_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (A_3)_{\text{orthorhombic}}$$

Hence the cubic system also satisfies the constraints of orthorhombic system. All the elastic modulli which are found to be zero in orthorhombic case is also zero here. Hence the number of independent modulli is

$$21 - \underbrace{12}_{\text{from orthorhombic case}} - \underbrace{6}_{\text{from (23) to (28) of this session}} = 3$$

3)Show that an isotropic system will have only two independent elastic moduli.

If the system is isotropic, then it is symmetric with respect to any axis of rotation. Let us consider a special case of rotation about 3-axis,45^o rotation. The corresponding A matrix will be

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Considering the transformation of C'_{66}, we get

$$C'_{66} = C'_{1212} = \sum a_{1p}a_{2q}a_{1r}a_{2s}C_{pqrs}$$

This transformation involves the coefficients a_{ij} only from 1^{st} and 2^{nd} row of A matrix. And as $a_{i3} = 0$ for $i = 1, 2$, this summation will have only 16 non-zero term. We can also see that the coefficient of C_{pqrs} in each of these terms is either $\frac{1}{4}$ (when $q = s$) or $-\frac{1}{4}$ (when $q \neq s$). And C_{pqrs} should also satisfy the symmetries of the cubic system for it to be isotropic. Keeping these things in mind, expanding the summation we get,

$$C'_{66} = \frac{1}{4} \times \begin{pmatrix} +C_{1111} - C_{1112} + C_{1121} - C_{1122} \\ -C_{1211} + C_{1212} - C_{1221} + C_{1222} \\ +C_{2111} - C_{2112} + C_{2121} - C_{2122} \\ -C_{2211} + C_{2212} - C_{2221} + C_{2222} \end{pmatrix}$$

$$C'_{66} = \frac{1}{4} \times \begin{pmatrix} +C_{11} - \cancel{C_{16}} + \cancel{C_{16}} - C_{12} \\ -C_{61} + \cancel{C_{66}} - \cancel{C_{66}} + C_{62} \\ +C_{61} - \cancel{C_{66}} + \cancel{C_{66}} - C_{62} \\ -C_{21} + \cancel{C_{26}} - \cancel{C_{26}} + C_{22} \end{pmatrix}$$

Now using cubic symmetry constraints,

$$C_{61} = 0; \quad C_{62}; \quad C_{21} = C_{12}; \quad C_{22} = C_{11}$$

Thus

$$C'_{66} = \frac{1}{4}(2C_{11} - 2C_{12}) = \frac{1}{2}(C_{11} - C_{12})$$

Thus by requiring symmetry we get

$$C_{66} = C_{55} = C_{44} = \frac{1}{2}(C_{11} - C_{12})$$

The above equation gives one more constraint on a cubic system. Thus for isotropy , number of independent of elastic modulli is $3 - 1 = 2$.