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Assignment 3.

→ Show that isotropic material have only two independent Elastic Moduli.

Soln:- we know

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

where C_{ijkl} is stiffness and it has 81 components in 9x9 Matrix notation.

* But from, homogeneous Eqb of body

$$\sigma_{ij} = \sigma_{ji} \quad \&$$

also $\epsilon_{ij} = \epsilon_{ji}$ (Symmetric matrix)

So, $\left[C_{ijkl} = C_{jikl} = C_{ijlk} \right]$

using it no. of independent component becomes 36.

* & Strain energy(ψ) is a scalar function

Elastic work done

$$d\psi = \sigma_{ij}\, d\epsilon_{ij} = C_{ijkl}\,\epsilon_{kl}\, d\epsilon_{ij}$$

$$\therefore \frac{d\varphi}{d\epsilon_{ij}} = C_{ijkl}\,\epsilon_{kl} = \sigma_{ij}$$

$$\text{or} \quad \frac{\partial}{\partial \epsilon_{kl}}\left[\frac{d\psi}{d\epsilon_{ij}}\right] = C_{ijkl}$$

$$\text{or} \quad \frac{d^2\psi}{\partial \epsilon_{kl}\,\partial \epsilon_{ij}} = C_{ijkl} = \frac{\partial}{\partial \epsilon_{ij}}\left[\frac{d\psi}{d\epsilon_{kl}}\right] = C_{klij}$$

So, $\left[C_{ijkl} = C_{klij} \right]$

It reduces no. of independent component from 36 to 21 only.

Which can be written in Voigt's notation like:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

where

$\sigma_1 = \sigma_{11}$ $\quad C_{11} = C_{1111}$ $\quad \epsilon_1 = \epsilon_{11}$

$\sigma_2 = \sigma_{22}$ $\quad C_{12} = C_{1122}$ $\quad \epsilon_2 = \epsilon_{22}$

$\sigma_3 = \sigma_{33}$ $\quad \ldots$ $\quad \epsilon_3 = \epsilon_{33}$

$\sigma_4 = \sigma_{23}$ $\quad \ldots$ $\quad \epsilon_4 = 2\epsilon_{23}$

$\sigma_5 = \sigma_{13}$ $\quad C_{34} = C_{3323}$ $\quad \epsilon_5 = 2\epsilon_{13}$

$\sigma_6 = \sigma_{12}$ $\quad$ So, on. $\quad \epsilon_6 = 2\epsilon_{12}$

We know that isotropic material have identical property in all direction. So, any rotation from any axis will not change its property in that direction.

So, we are applying a 4-fold symmetry operation about z-axis. then there should be.

$$C'_{ijkl} = C_{pqrs}$$

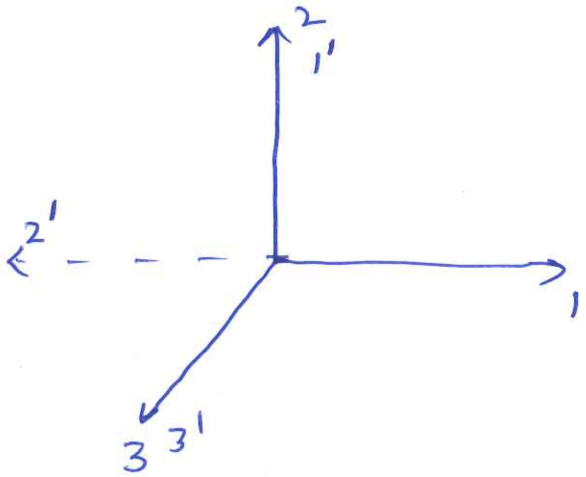
where C_{pqrs} is stiffness in old co-ordinate system.

& C'_{ijkl} is stiffness in new co-ordinate system.

in voigts notation it can be written as.

$$[C'_{mn} = C_{ab}]$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = a_{ij}$$



For ~~the~~ transformation.

$$C'_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, C_{pqrs}$$

It can be easily seen that after 4-fold rotation.

$1 \to 2$	or $11 \to 22$	in voigt's notation.
$2 \to -1$	$22 \to (-1)(-1) \to 11$	$1 \to 2$
$3 \to 3$	$33 \to 33$	$2 \to 1$
	$23 \to -13$	$3 \to 3$
	$13 \to 23$	$4 \to -5$
	$12 \to -12$	$5 \to 4$
		$6 \to -6$

So, component of stiffness in new co-ordinate system can be written as:-

(1) $C'_{11} \overset{\text{Transformation}}{=} C_{22} \overset{\text{Symmetry}}{=} C_{11}$

(2) $C'_{12} = C_{21} = C_{12}$

(3) $C'_{13} = C_{23} = C_{13}$

(4) $C'_{14} = -C_{25} = C_{14}$

(5) $C'_{15} = C_{24} = C_{15}$

(6) $C'_{16} = -C_{26} = C_{16}$

(7) $C'_{22} = C_{11} = C_{22}$

(8) $C'_{23} = C_{13} = C_{23}$

(9) $C'_{24} = -C_{15} = C_{24}$

(10) $C'_{25} = C_{14} = C_{25}$

(11) $C'_{26} = -C_{16} = C_{26}$

(12) $C'_{33} = C_{33} = C_{33}$

(13) $C'_{34} = -C_{35} = C_{34}$

(14) $C'_{35} = C_{34} = C_{35}$

(15) $C'_{36} = -C_{36} = C_{36}$

(16) $C'_{44} = C_{55} = C_{44}$

(17) $C'_{45} = -C_{54} = C_{45}$

(18) $C'_{46} = C_{56} = C_{46}$

(19) $C'_{55} = C_{44} = C_{55}$

(20) $C'_{56} = -C_{46} = C_{56}$

(21) $C'_{66} = C_{66} = C_{66}$

From (4) & (10)

$$C_{14} = -C_{25} = -C_{14} = 0$$

From (5) & (9)

$$C_{15} = C_{24} = -C_{15} = 0$$

From (13) & (14)

$$C_{34} = C_{35} = -C_{34} = 0$$

From (15) $C_{36} = -C_{36} = 0$

From (17)

$$C_{45} = -C_{45} = 0$$

From (18) & (20)

$$C_{46} = C_{56} = -C_{46} = 0$$

From (1)

$$C_{11} = C_{22}$$

From (3)

$$C_{23} = C_{13}$$

From (19)

$$C_{44} = C_{55}$$

From (11)

$$C_{16} = -C_{26}$$

So, we have now 10 components becomes 'zero' and 4 components are related to other. The no. of independent component reduces from 21 to 7 only.

which can be written as.

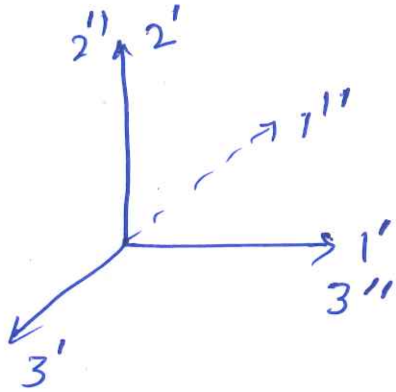
$$C'_{ijkl} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

→ We are again applying 4-fold operation about y-axis.

then $C''_{ijkl} = C'_{pqrs} = C_{efgh}$

where C''_{ijkl} is new co-ordinate system after rotation about y-axis.

or $C''_{mn} = C'_{ab} = C_{op}$



$$A = [a_{ij}] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

& $\left[C''_{ijkl} = a_{ip} a_{jq} a_{kr} a_{ls} C'_{pqrs} \right]$.

In voigt's notation (rotation of 90° about y-axis)

$1 \to -3$	$11 \to 33$	$1 \to 3$
$2 \to 2$	$22 \to 22$	$2 \to 2$
$3 \to 1$	$33 \to 11$	$3 \to 1$
	$23 \to 21$	$4 \to 6$
	$13 \to -31$	$5 \to -5$
	$12 \to -32$	$6 \to -4$

The component of stiffness in new co-ordinate system are :-

(22) $C''_{11} = C_{33} = C_{11}$

(23) $C''_{12} = C_{32} = C_{12}$

(24) $C''_{13} = C_{31} = C_{13}$

(25) $C''_{16} = -C_{34} = C_{16}$

(26) $C''_{22} = C_{22} = C_{22}$

(27) $C''_{23} = C_{21} = C_{23}$

(28) $C''_{26} = -C_{24} = C_{26}$

(29) $C''_{33} = C_{11} = C_{33}$

(30) $C''_{44} = C_{66} = C_{44}$

(31) $C''_{55} = C_{55} = C_{55}$

(32) $C''_{66} = C_{44} = C_{66}$

From (25), (13) & 14

$$C_{16} = -C_{34} = 0$$

From (28), (5) & 9

$$C_{26} = -C_{24} = 0$$

From (1) & (22)

$$C_{11} = C_{22} = C_{33}$$

From (19) & (32)

$$C_{44} = C_{55} = C_{66}$$

From (23)

$$C_{32} = C_{12} = C_{23}$$

So, Now No. of independent component reduces from 7 to only '3'. and can be written as

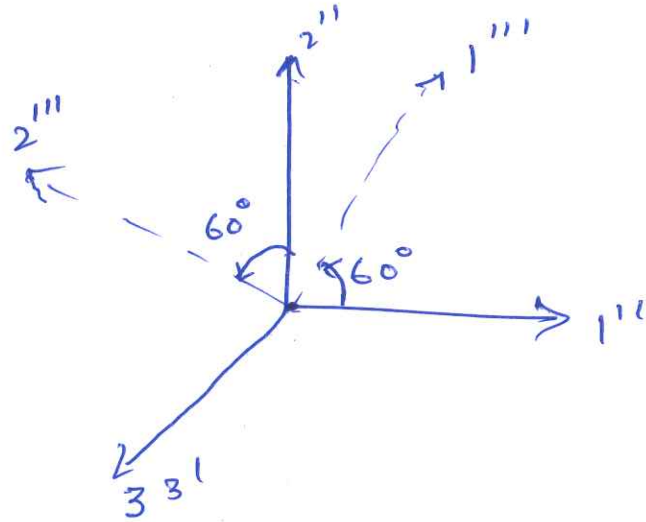
$$C''_{ijkl} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}$$

It expresses Cubical symmetry.

Since for isotropic material property will be same on any fold rotation. so Let's apply 6-fold rotation about Z-axis.

& $C'''_{ijkl} = C''_{pqrs}$

or $C'''_{mn} = C''_{ab}$



$$A = [a_{ij}] = \begin{bmatrix} \cos 60 & \cos 30 & 0 \\ \cos 150 & \cos 60 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[C'''_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, C''_{pqrs} \right].$$

In voigt's notation transformation of axes after 6-fold rotation can be written as:-

From 'A' matrix.

$$1 \rightarrow \frac{1}{2}(1) + \frac{\sqrt{3}}{2}(2)$$

$$2 \rightarrow -\frac{\sqrt{3}}{2}(1) + \frac{1}{2}(2)$$

$$3 \rightarrow 3$$

where (1) (2) denote old co-ordinate axes.

voigt's.

So; $1 \rightarrow 11 \rightarrow \left(\frac{1}{2}(1) + \frac{\sqrt{3}}{2}(2)\right)\left(\frac{1}{2}(1) + \frac{\sqrt{3}}{2}(2)\right)$

$$\rightarrow \frac{1}{4}(11) + \frac{\sqrt{3}}{2}(12) + \frac{3}{4}(22)$$

$$\rightarrow \frac{1}{4}(1) + \frac{\sqrt{3}}{2}(6) + \frac{3}{4}(2)$$

i.e. $\left[1 \rightarrow \frac{1}{4}(1) + \frac{\sqrt{3}}{2}(6) + \frac{3}{4}(2)\right]$ —— (A)

$$6 \rightarrow 12 \rightarrow \left(\frac{1}{2}(1) + \frac{\sqrt{3}}{2}(2)\right)\left(-\frac{\sqrt{3}}{2}(1) + \frac{1}{2}(2)\right)$$

$$\rightarrow -\frac{\sqrt{3}}{4}(11) - \frac{1}{2}(12) + \frac{\sqrt{3}}{4}(22)$$

i.e. $\left[6 \rightarrow -\frac{\sqrt{3}}{4}(1) - \frac{1}{2}(6) + \frac{\sqrt{3}}{4}(2)\right]$ —— (B)

$$4 \rightarrow 23 \rightarrow -\frac{\sqrt{3}}{2}(13) + \frac{1}{2}(23)$$

or $\left[4 \rightarrow -\frac{\sqrt{3}}{2}(5) + \frac{1}{2}(4)\right]$. —— (C)

Using (A) (B) (C) voigt's notation transformation,

$$C_{11}''' = \left[\frac{1}{4}(1) + \frac{\sqrt{3}}{2}(6) + \frac{3}{4}(2)\right]\left[\frac{1}{4}(1) + \frac{\sqrt{3}}{2}(6) + \frac{3}{4}(2)\right] C$$

$$= \frac{1}{16} C_{11} + \left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8}\right) C_{16} + \left(\frac{3}{16} + \frac{3}{16}\right) C_{12} + \frac{3}{4} C_{66}$$

$$+ \left(\frac{3\sqrt{3}}{8} + \frac{3\sqrt{3}}{8}\right) C_{62} + \frac{9}{16} C_{22}$$

or $C_{11}''' \overset{\text{symmetry.}}{=} C_{11} = \frac{1}{16} C_{11} + \frac{\sqrt{3}}{4} C_{16} + \frac{3}{8} C_{12} + \frac{3}{4} C_{66} + \frac{3\sqrt{3}}{4} C_{62}$

$$+ \frac{9}{16} C_{22}$$

& $C_{16} = 0$, $C_{62} = 0$ & $C_{66} = C_{44}$

$$\therefore C_{11} = \frac{1}{16} C_{11} + \frac{3}{8} C_{12} + \frac{3}{4} C_{44} + \frac{9}{16} C_{11}$$

or $\frac{16}{16} C_{11} - \frac{10}{16} C_{11} = \frac{6}{16} C_{12} + \frac{12}{16} C_{44}$

or $\frac{6}{16} C_{11} = \frac{6}{16} C_{12} + \frac{12}{16} C_{44}$

or $\left[C_{44} = \frac{C_{11} - C_{12}}{2} \right]$

So, No. of independent component of stiffness matrix has reduces from '3' to '2' Now. which can be written as :-

For Isotropic material :-

$$C_{ijkl} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ & & & & \frac{C_{11}-C_{12}}{2} & 0 \\ & & & & & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

The independent components are

C_{11} & C_{12}. only.

Proved