MT 253 : Mechanical Behaviour of Materials

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Extra Credit Home Work

0.5
Excellent

Q.) Show that an isotropic system will have only two elastic modulii.

Sol: Modulus tensor is generally represented as $\underset{\sim}{C}$ or $\underset{\sim}{S}$ based on

$\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\varepsilon}$ or $\underset{\sim}{\varepsilon} = \underset{\sim}{S}\,\underset{\sim}{\sigma}$

Consider,

$\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\varepsilon}$; where $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ are stress & strain tensors.

$\Rightarrow \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ [Einstein Notation]

So, the Modulus tensor is a fourth order tensor with $3^4 = 81$ modulus constants.

As, stress and strain are symmetric tensors,

we have, $\sigma_{ij} = \sigma_{ji}$

$\varepsilon_{kl} = \varepsilon_{lk}$

So, $C_{ijkl} = C_{jikl} = C_{jilk} = C_{ijlk} \rightarrow (1)$

- This is called Minor Symmetry.

Based on this, we can say that 81 elastic constants will turn out to be 36 elastic constants. (independent constants)

Similarly as strain energy is a state variable as a function of strain (i.e., strain-path invariant), we obtain

$C_{ijkl} = C_{klij}$ → Also called Major symmetry

Now 36 elastic constants ⟶ 21 elastic constants (independent constants)

- Since elastic moduli ~~are~~ a physical property of a material, it has to follow the crystal symmetry as per Neumann principle.

- For an isotropic material, every plane is a plane of symmetry.

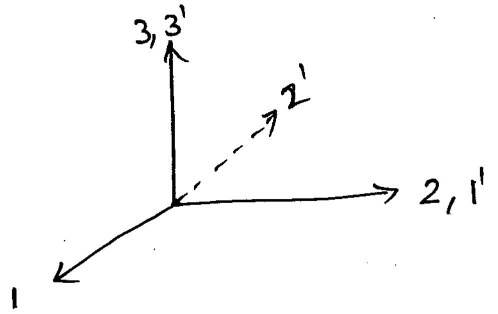
So, if we consider that 6×6 matrix obtained after minor symmetry argument & major symmetry argument, it will be of the form

→ 21 independent elastic constants

$$C_{ijkl} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{1123} & C_{2223} & C_{3323} & C_{2323} & C_{2313} & C_{2312} \\ C_{1113} & C_{2213} & C_{3313} & C_{2313} & C_{1313} & C_{1312} \\ C_{1112} & C_{2212} & C_{3312} & C_{2312} & C_{1312} & C_{1212} \end{bmatrix}$$

Now, for isotropic material, every plane is symmetric, So if we first apply 4 fold symmetry about the three axes, we result into 3 transformation matrices.

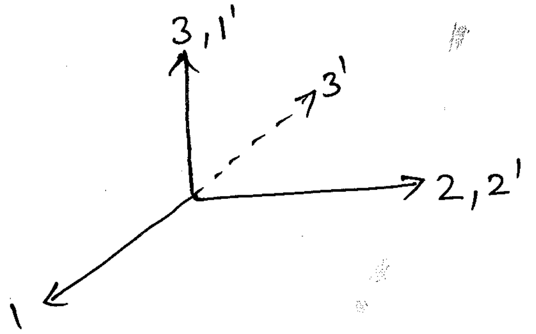
(i) 90° rotation about 3-axis



$$\underset{\sim}{A}_3 = \begin{pmatrix} \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \cos 180^\circ & \cos 90^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{pmatrix}$$

$$\boxed{\underset{\sim}{A}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

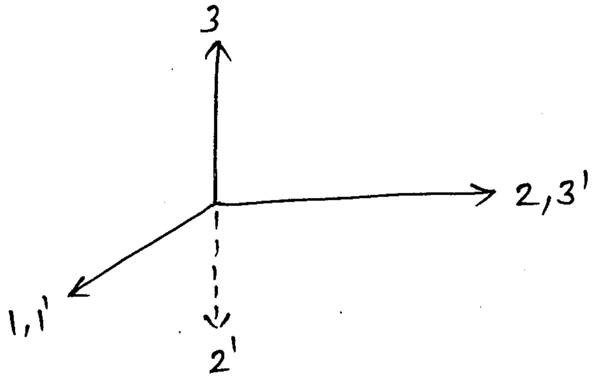
(ii) 90° rotation about 2-axis



$$\underset{\sim}{A}_2 = \begin{pmatrix} \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \\ \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \cos 180^\circ & \cos 90^\circ & \cos 90^\circ \end{pmatrix}$$

$$\boxed{\underset{\sim}{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}}$$

(iii) 90° rotation about 1-axis



$$\underset{\sim}{A}_1 = \begin{pmatrix} \cos 0^\circ & \cos 90^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 180^\circ \\ \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \end{pmatrix}$$

$$\boxed{\underset{\sim}{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}$$

As mentioned earlier, (Neumann's principle), we should get the same ~~symm~~ property or same value for elastic constant after 90° rotation (i.e., following a symmetry operation)

$$C'_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, C_{pqrs}$$

$$\Rightarrow \underbrace{C'_{1111} = a_{1p}\, a_{1q}\, a_{1r}\, a_{1s}\, C_{pqrs}}_{\text{from basis transformation.}} = \underbrace{C_{1111}}_{\text{from symmetry}}$$

$$= a_{1p}\, a_{1q}\, a_{1r}.\, a_{1s}\, C_{pqrs}$$

(i) If the 90° rotation is about 3-axis,

$$\underset{\sim}{A}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C'_{1111} = a_{1p}\, a_{1q}\, a_{1r} \left[\underbrace{a_{11}}_{0} C_{pqr1} + \underbrace{a_{12}}_{1} C_{pqr2} + \underbrace{a_{13}}_{0} C_{pqr3} \right]$$

$$= a_{1p}\, a_{1q} \left[a_{1r}\, C_{pqr2} \right]$$

$$= a_{1p}\, a_{1q} \left[\underbrace{a_{11}}_{0} C_{pq12} + \underbrace{a_{12}}_{1} C_{pq22} + \underbrace{a_{13}}_{0.} C_{pq32} \right]$$

$$= a_{1p} \left[a_{1q}\, C_{pq22} \right]$$

$$= a_{1p} \left[\underbrace{a_{11}}_{0} C_{p122} + \underbrace{a_{12}}_{1} C_{p222} + \underbrace{a_{13}}_{0} C_{p322} \right]$$

$$= a_{1p}\, C_{p222} = \underbrace{a_{11}}_{0} C_{1222} + \underbrace{a_{12}}_{1} C_{2222} + \underbrace{a_{13}}_{0} C_{3222} = C_{2222}$$

$\boxed{C'_{1111} = C_{2222} = C_{1111}}$; This reduces the no. of constants by 1 (ie., 21 → 20)

Now, consider

(ii) C_{1111} 90° rotation around 2-axis where $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$C'_{1111} = a_{1p} a_{1q} a_{1r} a_{1s} C_{pqrs} = C_{1111}$$
$$= a_{1p} a_{1q} a_{1r} C_{pqr3}$$
$$= a_{1p} a_{1q} C_{pq33}$$
$$= a_{1p} C_{p333}$$
$$= C_{3333}$$

$\boxed{C'_{1111} = C_{3333} = C_{1111}}$; This reduces no. of constants by 1 (20 → 19)

Now, consider C_{1122}

(i) C_{1122} : 90° rotation around 2-axis where $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$C'_{1122} = a_{1p} a_{1q} a_{2r} a_{2s} C_{pqrs} = C_{1122}$$
$$= a_{1p} a_{1q} a_{2r} C_{pqr2}$$
$$= a_{1p} a_{1q} C_{pq22}$$
$$= a_{1p} C_{p322}$$
$$= C_{3322}$$
$$= C_{2233} \text{ [from minor symmetry]}$$

$\boxed{C'_{1122} = C_{2233} = C_{1122}}$; 19 → 18

(ii) C_{1122} : 90° rotation around 1-axis where $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$C'_{1122} = a_{1p} a_{1q} a_{2r} a_{2s} C_{pqrs} = C_{1122}$$
$$= a_{1p} a_{1q} a_{2r} (-C_{pqr3})$$
$$= a_{1p} a_{1q} (C_{pq33})$$
$$= a_{1p} (C_{p133}) = C_{1133}$$

$\boxed{C'_{1122} = C_{1133} = C_{1122}}$

18 → 17

Now, consider C_{1123}

(i) C_{1123} : 90° rotation around 3-axis where $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$C'_{1123} = a_{1p} a_{1q} a_{2r} a_{3s} C_{pqrs} = C_{1123}$$
$$= a_{1p} a_{1q} a_{2r} C_{pqr3}$$
$$= a_{1p} a_{1q} [-C_{pq13}]$$
$$= a_{1p} [-C_{p213}]$$
$$= -C_{2213}.$$

$$\boxed{C'_{1123} = -C_{2213} = C_{1123}} \quad ; \quad 17 \rightarrow 16.$$

(ii) C_{1123}: 90° rotation around 2-axis where $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$C'_{1123} = a_{1p} a_{1q} a_{2r} a_{3s} C_{pqrs} = C_{1123}$$
$$= a_{1p} a_{1q} a_{2r} [-C_{pqr1}]$$
$$= a_{1p} a_{1q} [-C_{pq21}]$$
$$= a_{1p} [-C_{p321}]$$
$$= -C_{3321} = -C_{3312}$$

$$\boxed{C'_{1123} = -C_{3312} = C_{1123}} \quad ; \quad 16 \rightarrow 15$$

(iii) C_{1123} : 90° rotation around 1-axis where $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$C'_{1123} = a_{1p} a_{1q} a_{2r} a_{3s} C_{pqrs} = C_{1123}$$
$$= a_{1p} a_{1q} a_{2r} [C_{pqr2}]$$
$$= a_{1p} a_{1q} [-C_{pq32}]$$
$$= a_{1p} [-C_{p132}] = -C_{1132} = -C_{1123} = C_{1123}.$$

$$\therefore \boxed{C_{1123} = C_{2213} = C_{3312} = 0} \quad ; \quad 15 \rightarrow 14 \quad \rightarrow (2)$$

Now, consider C_{1113}

(i) C_{1113} : 90° rotation about 3-axis where $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$C'_{1113} = a_{1p} a_{1q} a_{1r} a_{3s} C_{pqrs} = C_{1113}$$
$$= a_{1p} a_{1q} a_{1r} [C_{pqr3}]$$
$$= a_{1p} a_{1q} (C_{pq23})$$
$$= a_{1p} (C_{p223})$$
$$= C_{2223}$$

$\boxed{C'_{1113} = C_{2223} = C_{1113}}$; $14 \to 13$

(ii) C_{1113} : 90° rotation about 2-axis where $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$C'_{1113} = a_{1p} a_{1q} a_{1r} a_{3s} C_{pqrs} = C_{1113}$$
$$= a_{1p} a_{1q} a_{1r} [-C_{pqr1}]$$
$$= a_{1p} a_{1q} [-C_{pq31}]$$
$$= a_{1p} [-C_{p331}]$$
$$= -C_{3331} = -C_{3313} \text{ (symmetry)}$$

$\boxed{C'_{1113} = -C_{3313} = C_{1113}}$; $13 \to 12$

(iii) C_{1113} : 90° rotation about 1-axis where $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$C'_{1113} = a_{1p} a_{1q} a_{1r} a_{3s} C_{pqrs} = C_{1113}$$
$$= a_{1p} a_{1q} a_{1r} [C_{pqr2}]$$
$$= a_{1p} a_{1q} [C_{pq12}]$$
$$= a_{1p} [C_{p112}]$$
$$= C_{1112}$$

$\boxed{C'_{1113} = C_{1112} = C_{1113}}$; $12 \to 11$

$\therefore$ $\boxed{C_{1113} = -C_{3313} = C_{2223} = C_{1112}}$

Now, consider C_{1112}

(i) C_{1112} : 90° rotation about 3-axis where $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$C'_{1112} = a_{1p} a_{1q} a_{1r} a_{2s} C_{pqrs} = C_{1112}$$
$$= a_{1p} a_{1q} a_{1r} [-C_{pqr1}]$$
$$= a_{1p} a_{1q} [-C_{pq21}]$$
$$= a_{1q} [-C_{p221}]$$
$$= -C_{2221} = -C_{2212} \text{ (symmetry)}$$

$$\boxed{C'_{1112} = -C_{2212} = C_{1112}}$$ $11 \to 10.$

$\therefore$ $C_{1113} = C_{1113} = -C_{3313} = C_{2223} = -C_{2212}$

(ii) C_{1112} : 90° rotation about 2-axis where $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$C'_{1112} = a_{1p} a_{1q} a_{1r} a_{2s} C_{pqrs} = C_{1112}$$
$$= a_{1p} a_{1q} a_{1r} [C_{pqr2}]$$
$$= a_{1p} a_{1q} [C_{pq32}]$$
$$= a_{1p} [C_{p332}]$$
$$= C_{3332} = C_{3323} \text{ (symmetry)}$$

$$\boxed{C'_{1112} = C_{3323} = C_{1112}}$$ $10 \to 9$

$$C_{1112} = C_{3323} = -C_{2212} = C_{1113} = -C_{3313} = C_{2223}$$

(iii) C_{1112} : 90° rotation about 1-axis where $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$C'_{1112} = a_{1p} a_{1q} a_{1r} a_{2s} C_{pqrs} = C_{1112}$$
$$= a_{1p} a_{1q} a_{1r} [-C_{pqr3}]$$
$$= a_{1p} a_{1q} [-C_{pq13}]$$
$$= a_{1p} [-C_{p113}]$$
$$= -C_{1113} = C_{1112}$$

But initially, $C_{1112} = C_{1113}$ [When C_{1113}: 90° rotation about 1-axis]

$\therefore \boxed{C_{1112} = C_{1113} = 0}$ $9 \to 8$

So, $\boxed{C_{1112} = C_{1113} = C_{3323} = C_{2212} = C_{3313} = C_{2223} = 0} \rightarrow (3)$

From (2), $C_{1123} = C_{2213} = C_{3312} = 0$

~~So, the~~

Now, consider C_{2313}

(i) C_{2313}: 90° rotation about 3-axis where $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$
\begin{aligned}
C'_{2313} &= a_{2p} a_{3q} a_{1r} a_{3s} C_{pqrs} = C_{2313} \\
&= a_{2p} a_{3q} a_{1r} [C_{pqr3}] \\
&= a_{2p} a_{3q} [C_{pq23}] \\
&= a_{2p} [C_{p323}]
\end{aligned}
$$

$$C'_{2313} = -C_{1323} = \underset{\text{(symmetry)}}{-C_{2313}} = C_{2313}$$

$\therefore \boxed{C_{2313} = \cancel{C_{1323}} = 0}$ $\quad 8 \rightarrow 7$

(ii) C_{2313}: 90° rotation about 2-axis where $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$
\begin{aligned}
C'_{2313} &= a_{2p} a_{3q} a_{1r} a_{3s} C_{pqrs} = C_{2313} \\
&= a_{2p} a_{3q} a_{1r} [-C_{pqr1}] \\
&= a_{2p} a_{3q} [-C_{pq31}] \\
&= a_{2p} [C_{p131}] \\
&= C_{2131} = C_{1213} \text{ [symmetry]} = C_{2313}
\end{aligned}
$$

$\boxed{C_{2313} = C_{1213} = 0}$ $\quad 7 \rightarrow 6.$

(iii) C_{2313}: 90° rotation about 1-axis where $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

$$
\begin{aligned}
C'_{2313} &= a_{2p} a_{3q} a_{1r} a_{3s} C_{pqrs} = C_{2313} \\
&= a_{2p} a_{3q} a_{1r} [C_{pqr2}] \\
&= a_{2p} a_{3q} [C_{pq12}] \\
&= a_{2p} [C_{p212}] = -C_{3213} = \underset{\text{(Symmetry)}}{-C_{2312}} = C_{2313}
\end{aligned}
$$

$\boxed{C_{2313} = C_{2312} = C_{1213} = 0}$ $\quad 6 \rightarrow 5$

Now, consider C_{2323}

(i) C_{2323} : 90° rotation about 3-axis where $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$C'_{2323} = a_{2p}\, a_{3q}\, a_{2r}\, a_{3s}\, C_{pqrs} = C_{2323}$$
$$= a_{2p}\, a_{3q}\, a_{2r}\left[C_{pqr3}\right]$$
$$= a_{2p}\, a_{3q}\left[-C_{pq13}\right]$$
$$= a_{2p}\left[-C_{p313}\right]$$
$$= C_{1313} = C_{2323}$$

$$\boxed{C'_{2323} = C_{1313} = C_{2323}} \qquad 5 \rightarrow 4$$

(ii) C_{2323} : 90° rotation about 2-axis where $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

$$C'_{2323} = a_{2p}\, a_{3q}\, a_{2r}\, a_{3s}\, C_{pqrs} = C_{2323}$$
$$= a_{2p}\, a_{3q}\, a_{2r}\left[-C_{pqr1}\right]$$
$$= a_{2p}\, a_{3q}\left[-C_{pq21}\right]$$
$$= a_{2p}\left[C_{p121}\right]$$
$$= C_{2121} = C_{1212} \text{ (symmetry)}$$

$$\boxed{C_{2323} = C_{1313} = C_{1212}}$$

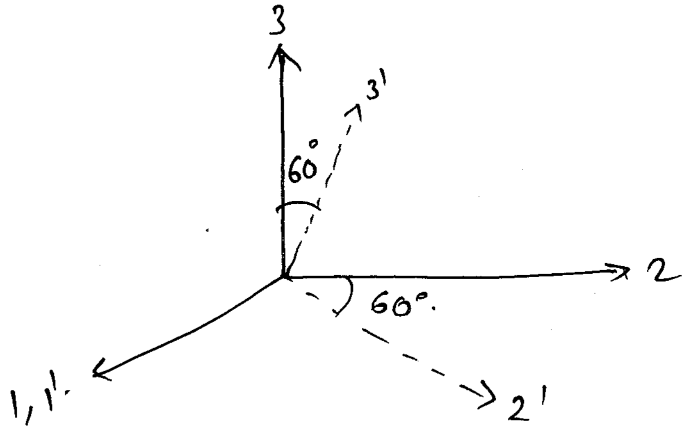
$$\boxed{C'_{2323} = C_{1212} = C_{2323}} \qquad 4 \rightarrow 3$$

Now the matrix turns out to be

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1122} & C_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2323} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{2323} \end{bmatrix}$$

C_{1111}, C_{1122}, C_{2323} are 3 independent elastic constants after 4-fold rotation about the three axis.

Now, if we consider 6 fold rotation of C_{2323} about 1-axis.



The transformation matrix

$$= \begin{pmatrix} \cos 0^\circ & \cos 90^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 60^\circ & \cos 150^\circ \\ \cos 90^\circ & \cos 30^\circ & \cos 60^\circ \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Consider C_{2323} rotation about 1-axis [6-fold rotation]

$$C'_{2323} = a_{2p}\, a_{3q}\, a_{2r}\, a_{3s}\, C_{pqrs} = C_{2323}$$

$$= a_{2p}\, a_{3q}\, a_{2r}\left[-\frac{\sqrt{3}}{2} C_{pqr1} + \frac{1}{2} C_{pqr3}\right]$$

$$= a_{2p}\, a_{3q}\left[-\frac{\sqrt{3}}{2} C_{pq21} + \frac{1}{2} C_{pq23}\right]$$

$$= a_{2p}\left[-\frac{\sqrt{3}}{2}\left[-\frac{\sqrt{3}}{2} C_{p121} + \frac{1}{2} C_{p123}\right] + \frac{1}{2}\left[-\frac{\sqrt{3}}{2} C_{p321} + \frac{1}{2} C_{p323}\right]\right]$$

$$= a_{2p}\left[\frac{3}{4} C\right.$$

$$C'_{2323} = a_{2p}\, a_{3q}\, a_{2r}\, a_{3s}\, C_{pqrs} = C_{2323}$$

$$= a_{2p}\, a_{3q}\, a_{2r}\left[\frac{\sqrt{3}}{2} C_{pqr2} + \frac{1}{2} C_{pqr3}\right]$$

$$= a_{2p}\, a_{3q}\left[\frac{1}{2}\left[\frac{\sqrt{3}}{2} C_{pq22} + \frac{1}{2} C_{pq23}\right] - \frac{\sqrt{3}}{2}\left[\frac{\sqrt{3}}{2} C_{pq32} + \frac{1}{2} C_{pq33}\right]\right]$$

$$= a_{2p}\, a_{3q}\left[\frac{\sqrt{3}}{4} C_{pq22} + \frac{1}{4} C_{pq23} - \frac{3}{4} C_{pq32} - \frac{\sqrt{3}}{4} C_{pq33}\right]$$

$$= a_{2p}\left[\frac{\sqrt{3}}{2}\left[\frac{\sqrt{3}}{4} C_{p222} + \frac{1}{4} C_{p223} - \frac{3}{4} C_{p232} - \frac{\sqrt{3}}{4} C_{p233}\right]\right.$$

$$\left. + \frac{1}{2}\left[\frac{\sqrt{3}}{4} C_{p322} + \frac{1}{4} C_{p323} - \frac{3}{4} C_{p332} - \frac{\sqrt{3}}{4} C_{p333}\right]\right]$$

$$= a_{2p}\left[\frac{3}{8} C_{p222} + \frac{\sqrt{3}}{8} C_{p223} - \frac{3\sqrt{3}}{8} C_{p232} - \frac{3}{8} C_{p233} + \frac{\sqrt{3}}{8} C_{p322} + \frac{1}{8} C_{p323} - \frac{3}{8} C_{p332} - \frac{\sqrt{3}}{8} C_{p333} \right]$$

$$= \frac{1}{2}\left[\frac{3}{8} C_{2222} + \frac{\sqrt{3}}{8} C_{2223} - \frac{3\sqrt{3}}{8} C_{2232} - \frac{3}{8} C_{2233} + \frac{\sqrt{3}}{8} C_{2322} + \frac{1}{8} C_{2323} - \frac{3}{8} C_{2332} - \frac{\sqrt{3}}{8} C_{2333} \right]$$

$$- \frac{\sqrt{3}}{2}\left[\frac{3}{8} C_{3222} + \frac{\sqrt{3}}{8} C_{3223} - \frac{3\sqrt{3}}{8} C_{3232} - \frac{3}{8} C_{3233} + \frac{\sqrt{3}}{8} C_{3322} + \frac{1}{8} C_{3323} - \frac{3}{8} C_{3332} - \frac{\sqrt{3}}{8} C_{3333} \right]$$

$$= \frac{3}{16} C_{2222} + \frac{\sqrt{3}}{16} \cancelto{0}{C_{2223}} - \frac{3\sqrt{3}}{16} \cancelto{0}{C_{2232}} - \frac{3}{16} C_{2233} + \frac{\sqrt{3}}{16} \cancelto{0}{C_{2322}} + \frac{1}{16} C_{2323} - \frac{3}{16} C_{2332} - \frac{\sqrt{3}}{16} \cancelto{0}{C_{2333}} - \frac{3\sqrt{3}}{16} \cancelto{0}{C_{3222}} - \frac{3}{16} C_{3223} + \frac{9}{16} C_{3232} + \frac{3\sqrt{3}}{16} \cancelto{0}{C_{3233}} - \frac{3}{16} C_{3322} - \frac{\sqrt{3}}{16} \cancelto{0}{C_{3323}} + \frac{3\sqrt{3}}{16} \cancelto{0}{C_{3332}} + \frac{3}{16} C_{3333}$$

$$= \frac{3}{16}\left[C_{2222} + C_{3333} \right] - \frac{3}{16}\left[C_{2233} + C_{3322} \right] + \frac{1}{16}\left[C_{2323} - 3C_{2332} - 3C_{3223} + 9C_{3232} \right]$$

$$= \frac{3}{16}\left[2C_{1111} \right] - \frac{3}{16}\left[2C_{1122} \right] + \frac{1}{16}\left[4C_{2323} \right]$$

$$= \frac{3}{8} C_{1111} - \frac{3}{8} C_{1122} + \frac{1}{4} C_{2323}. \quad = C_{2323}$$

$$\Rightarrow \frac{3}{8} C_{1111} - \frac{3}{8} C_{1122} = \frac{3}{4} C_{2323}$$

$$\Rightarrow \frac{3}{8}\left[C_{1111} - C_{1122} \right] = \frac{3}{4}\left(C_{2323} \right)$$

$$\Rightarrow \frac{C_{1111} - C_{1122}}{2} = C_{2323}$$

$$\therefore \quad \boxed{C_{2323} = \frac{1}{2}\left(C_{1111} - C_{1122}\right)} \Rightarrow \underset{\substack{\text{elastic} \\ \text{constants}}}{3} \rightarrow \underset{\substack{\text{elastic} \\ \text{constants}}}{2}$$

The matrix now turns out to be

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{1122} & C_{1111} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left[C_{1111} - C_{1122}\right] & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\left[C_{1111} - C_{1122}\right] & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left[C_{1111} - C_{1122}\right] \end{bmatrix}$$

Thus, the isotropic materials have only two independent elastic modulii. viz., C_{1111} & C_{1122}.