Q. Construct the mohr circle by the following equation

$$\sigma' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Given equation,

$$\sigma' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (\sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta) & (-\sigma_{xx}\sin\theta + \sigma_{xy}\cos\theta) & 0 \\ (\sigma_{xy}\cos\theta + \sigma_{yy}\sin\theta) & (-\sigma_{xy}\sin\theta + \sigma_{yy}\cos\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\sigma_{xx}\cos^2\theta + \sigma_{xy}\cos\theta\sin\theta + \sigma_{xy}\cos\theta\sin\theta + \sigma_{yy}\sin^2\theta\right) & \left(-\sigma_{xx}\cos\theta\sin\theta + \sigma_{xy}\cos^2\theta - \sigma_{xy}\sin^2\theta + \sigma_{yy}\cos\theta\sin\theta\right) & 0 \\ \left(-\sigma_{xx}\sin\theta\cos\theta - \sigma_{xy}\sin^2\theta + \sigma_{xy}\cos^2\theta + \sigma_{yy}\sin\theta\cos\theta\right) & \left(\sigma_{xx}\sin^2\theta - \sigma_{xy}\sin\theta\cos\theta - \sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\cos\theta\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & 0 \\ \sigma'_{yx} & \sigma'_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(Let)}$$

Now;

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{xy}\cos\theta\sin\theta + \sigma_{xy}\cos\theta\sin\theta + \sigma_{yy}\sin^2\theta$$

$$= \sigma_{xx}\cos^2\theta + 2\sigma_{xy}\cos\theta\sin\theta + \sigma_{yy}\sin^2\theta$$

$\because \cos 2\phi = \cos^2\phi - \sin^2\phi = 2\cos^2\phi - 1 = 1 - 2\sin^2\phi$

and; $\sin 2\phi = 2\sin\phi\cos\phi$

$$= \sigma_{xx}\cdot\frac{1+\cos 2\theta}{2} + \sigma_{xy}\sin 2\theta + \sigma_{yy}\frac{1-\cos 2\theta}{2}$$

$$\boxed{\sigma'_{xx} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta} \quad \text{—— (I)}$$

$$\because \sigma'_{yy} = \sigma_{xx}\sin^2\theta - \sigma_{xy}\sin\theta\cos\theta - \sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\cos^2\theta$$

$$= \sigma_{xx}\frac{1-\cos 2\theta}{2} - \sigma_{xy}\sin 2\theta + \sigma_{yy}\frac{1+\cos 2\theta}{2}$$

$$\therefore \boxed{\sigma'_{yy} = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2} - \sigma_{xy}\sin^2\theta} \quad \text{—— (II)}$$

and, as

$$\sigma'_{xy} = \sigma'_{yx} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{xy}\cos^2\theta - \sigma_{xy}\sin^2\theta + \sigma_{yy}\cos\theta\sin\theta$$

$$= \frac{\sigma_{yy}-\sigma_{xx}}{2}\cdot 2\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\therefore \boxed{\sigma'_{xy} = \sigma_{xy}\cos 2\theta - \frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta} \quad \text{—— (III)}$$

P.T.O.

Now from equation ①,

$$\sigma'_{xx} - \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta \quad \text{——— (IV)}$$

squaring equation (IV) and (III), and on adding; we have

$$\Rightarrow \left(\sigma'_{xx} - \frac{\sigma_{xx}+\sigma_{yy}}{2}\right)^2 + (\sigma'_{xy})^2 = \left[\left\{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + (\sigma_{xy})^2\right\}^{1/2}\right]^2.$$

The above equation present the equation of the circle; if we consider σ'_{xx} and σ'_{xy} as variable; and where σ'_{xy} depands on variable σ'_{xx}.

So, Now we can take σ'_{xy} as y-axis and σ'_{xx} as x-axis.

co-ordinate of centre of circle $= \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$

and radius of the circle $= \left[\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + (\sigma_{xy})^2\right]^{1/2}$